

# CONFORMAL BLOCKS AND PAINLEVÉ FUNCTIONS

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ABSTRACT. This paper is based on my presentation at RIMS workshop on “Theory of Integrable Systems and Its Applications in Various Fields” held in Kyoto on 19–21, August 2015. The aim of the present paper is to give a short account of recent studies on relations between conformal blocks in the two dimensional conformal field theory and Painlevé functions. In addition, we present a conjecture on a combinatorial expansion formula of the three-point irregular conformal block at an irregular singular point, with two regular singular points and one irregular singular point. Our conjectural expansion formula is written in terms of pairs of skew Young diagrams, while the four-point regular conformal block, by AGT correspondence, is written in terms of pairs of Young diagrams.

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## 1. INTRODUCTION

**1.1. Painlevé equations.** Aiming to obtain new special functions, P. Painlevé classified second order nonlinear ordinary differential equations

$$R\left(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right) = 0$$

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*Key words and phrases.* conformal blocks, Painlevé function, Fourier expansion.

whose movable singular points are pole only and obtained new six equations around the beginning of the twentieth century, which are now called Painlevé equations:

$$\begin{aligned}
\text{P}_I \quad & y'' = 6y^2 + t, \\
\text{P}_{II} \quad & y'' = 2y^3 + ty + \alpha, \\
\text{P}_{III} \quad & y'' = \frac{1}{y}y'^2 - \frac{1}{y}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}, \\
\text{P}_{IV} \quad & y'' = \frac{1}{2y}y'^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}, \\
\text{P}_V \quad & y'' = \left( \frac{1}{2y} + \frac{1}{y-1} \right) y'^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2} \left( \alpha y + \frac{\beta}{y} \right) \\
& \quad + \gamma \frac{y}{t} + \delta \frac{y(y+1)}{y-1}, \\
\text{P}_{VI} \quad & y'' = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) y'^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) y' \\
& \quad + \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left( \alpha + \beta \frac{t}{y^2} + \gamma \frac{t-1}{(y-1)^2} + \delta \frac{t(t-1)}{(y-t)^2} \right).
\end{aligned}$$

Let us see a relation between Weierstrass  $\wp$  function and  $\text{P}_I$ . Replacing  $t$  of  $\text{P}_I$

$$\frac{d^2y}{dt^2} = 6y^2 + t$$

with  $-g_2/2 \in \mathbb{C}$ , we obtain a second-order differential equation

$$(1) \quad \frac{d^2y}{dt^2} = 6y^2 - \frac{1}{2}g_2,$$

which is derived by differentiating the differential equation

$$\left( \frac{dy}{dt} \right)^2 = 4y^3 - g_2y - g_3.$$

Hence, Weierstrass  $\wp$  function is a solution to (1). Weierstrass  $\sigma$  function is defined by

$$\wp = -\frac{d^2}{dt^2} \log \sigma.$$

The first Painlevé equation is

$$\frac{d^2y}{dt^2} = 6y^2 + t.$$

For any solution  $\lambda(t)$ , define  $\tau(t)$  by

$$\lambda(t) = -\frac{d^2}{dt^2} \log \tau(t).$$

For the other Painlevé equations, the tau functions are defined in the same way. As theta functions play an important role in the theory of elliptic functions, the tau functions  $\tau(t)$  are

relevant tools for the studies of the Painlevé functions. We remark that  $P_I$  is a Hamiltonian system with

$$H(t) = \frac{1}{2}\mu(t)^2 - 2\lambda(t)^3 - t\lambda(t) \quad \left( \mu(t) = \frac{d\lambda(t)}{dt} \right).$$

Note that  $H(t) = (\log \tau(t))'$ .  $H(t)$  satisfies a non-linear differential equation

$$\frac{d^3 H}{dt^3} + 6 \left( \frac{dH}{dt} \right)^2 + t = 0.$$

Conversely, a function  $H(t)$  satisfying the differential equation above defines a solution  $\lambda(t)$  to the first Painlevé equation.

**1.2. Fourier expansion.** For a long time, it had not been known an explicit series representation of the tau functions of Painlevé equations, until a remarkable discovery by Gamayun, Iorgov and Lisovyy [13]. They found that the tau function of the sixth Painlevé equation is a Fourier expansion in terms of the four point Virasoro conformal block with central charge  $c = 1$ . Conformal blocks are the building blocks of correlation functions of the two dimensional conformal field theory, which are defined as expectation values of vertex operators.

Their formula reads as

$$(2) \quad \tau_{VI}(t) = \sum_{n \in \mathbb{Z}} s^n C \left( \begin{matrix} \theta_1, \theta_t \\ \theta_\infty, \sigma + n, \theta_0 \end{matrix} \right) \mathcal{F} \left( \begin{matrix} \theta_1, \theta_t \\ \theta_\infty, \sigma + n, \theta_0 \end{matrix}; t \right),$$

where  $s, \sigma \in \mathbb{C}$ ,  $\mathcal{F}(\theta, \sigma; t) = t^{\sigma^2 - \theta_t^2 - \theta_0^2} (1 + O(t))$  is the four-point Virasoro conformal block with  $c = 1$ , and

$$C(\theta, \sigma) = \frac{\prod_{\epsilon, \epsilon' = \pm} G(1 + \theta_t + \epsilon \theta_0 + \epsilon' \sigma) G(1 + \theta_1 + \epsilon \theta_\infty + \epsilon' \sigma)}{\prod_{\epsilon = \pm} G(1 + 2\epsilon \sigma)},$$

where  $G(z)$  is the Barnes G-function such that  $G(z+1) = \Gamma(z)G(z)$ . By AGT correspondence, the four-point Virasoro conformal block has an explicit series representation (6).

Before this discovery, it was known by Jimbo [17] that the tau function of the sixth Painlevé equation is expanded as

$$\begin{aligned} \tau(t) = & \text{const. } t^{(\sigma^2 - \theta_0^2 - \theta_t^2)} \\ & \times \left( 1 + \frac{(\theta_0^2 - \theta_t^2 - \sigma^2)(\theta_\infty^2 - \theta_1^2 - \sigma^2)}{2\sigma^2} t \right. \\ & - \sum_{\epsilon = \pm} \frac{\hat{s}^\epsilon}{8\sigma^2(1 + 4\sigma^2)^2} (\theta_0^2 - (\theta_t - \epsilon\sigma)^2)(\theta_\infty^2 - (\theta_1 - \epsilon\sigma)^2) t^{1+2\epsilon\sigma} \\ & \left. + \sum_{j=2}^{\infty} \sum_{|k| \leq j} a_{jk} t^{j-2k\sigma} \right), \end{aligned}$$

where  $\hat{s}, \sigma$  are expressed by the monodromy data of a linear equation associated with  $P_{VI}$ . The first part of  $\tau(t)$ :

$$t^{\sigma^2 - \theta_0^2 - \theta_t^2} \left( 1 + \frac{(\theta_0^2 - \theta_t^2 - \sigma^2)(\theta_\infty^2 - \theta_1^2 - \sigma^2)}{2\sigma^2} t \right)$$

resembles the first part of the four-point conformal block of Virasoro CFT:

$$\begin{aligned} & \langle \Delta_4 | \cdot (\Phi_{\Delta_4, \Delta}^{\Delta_3}(1) \circ \Phi_{\Delta, \Delta_1}^{\Delta_2}(t) | \Delta_1 \rangle) \\ &= t^{\Delta - \Delta_2 - \Delta_1} \left( 1 + \frac{(\Delta + \Delta_3 - \Delta_4)(\Delta + \Delta_2 - \Delta_1)}{2\Delta} t + \sum_{k=2}^{\infty} c_k \left( \frac{z_2}{z_3} \right)^k \right). \end{aligned}$$

The definitions of a vertex operator and a conformal block are given in Section 2. Note that when  $c = 1$ , parametrizing  $\Delta_i = \theta_i^2$  is standard. It seems that this had been missed for a long time. Finally, it was noticed and led them to the discovery.

Proofs for the Fourier expansion of the tau function of  $P_{VI}$  in terms of the four-point conformal block were given in [16] by constructing a fundamental solution of a linear equation associated with  $P_{VI}$  using the conformal field theory, and [5] by showing that the Fourier expansion satisfies the bilinear equation for  $\tau_{VI}$  using embedding the direct sum of two Virasoro algebras into the super Virasoro algebra.

**1.3. Irregular case.** It is well known that Painlevé equations admit the following degeneration scheme:

$$\begin{array}{ccccccc} P_{VI} & \longrightarrow & P_V & \longrightarrow & P_{III} & \longrightarrow & P_{III}^{D_7} & \longrightarrow & P_{III}^{D_8} \\ & & & \searrow & & \searrow & & \searrow & \\ & & & & P_{IV} & \longrightarrow & P_{II} & \longrightarrow & P_I \end{array}$$

Looking at the Painlevé equations, we know the fixed singular points are as in Table 1. Here,

TABLE 1.

	singular points
$P_{VI}$	$0, 1, \infty$
$P_V, P_{III}, P_{III}^{D_7}, P_{III}^{D_8}$	$0, \infty$
$P_{IV}, P_{II}, P_I$	$\infty$

$\infty$  is an irregular singular point for Painlevé equations except  $P_{VI}$ , and 0 is a regular singular point for  $P_{VI}, P_V, P_{III}, P_{III}^{D_7}$  and  $P_{III}^{D_8}$ . In general, it is easier to compute confluent process of a series expansion at a regular singular point than at an irregular singular point. This is true for our case. Series expansions at the regular singular point 0 of the tau functions of  $P_V, P_{III}, P_{III}^{D_7}$  and  $P_{III}^{D_8}$  were obtained in [14] by taking some collision limits. It has not been reported at the moment that expansions at the irregular singular point  $\infty$  of the tau functions of Painlevé equations are obtained by some confluent process. Instead, it was conjectured that a Fourier expansion at  $\infty$  of the tau function for  $P_{III}^{D_8}$  can be computed recursively and the first few terms were obtained in [18].

It is natural to expect that series expansions at  $\infty$  of the tau functions are Fourier expansions in terms of irregular conformal blocks. Since we lacked enough knowledge on irregular conformal blocks, Fourier expansions of the tau functions in terms of irregular conformal blocks were not given. We need to understand expansions of irregular conformal blocks at irregular singular points more clearly. From this point of view, a precise definition of irregular vertex operators were introduced and irregular conformal blocks were given as expectation values of irregular

vertex operators [24]. Using newly introduced irregular conformal blocks, the author was able to obtain conjectural formulas for Fourier expansions at  $\infty$  of the tau functions of  $P_V$  and  $P_{IV}$  [24].

The remainder of this paper is organized as follows. In Section 2, a short review on regular conformal blocks are given. In Section 3, we recall what was known about irregular conformal blocks before [24]. In Section 4, a direct approach for obtaining irregular conformal blocks are explained, following [24]. In Section 5, we present conjectural formulas for Fourier expansions at  $\infty$  of the tau functions of  $P_V$  and  $P_{IV}$  proposed in [24] and give comments on a combinatorial formula for three-point irregular conformal block with two regular singular points and one irregular singular point of rank 1.

## 2. REGULAR CONFORMAL BLOCKS

The Virasoro algebra

$$\text{Vir} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}C$$

is the Lie algebra with commutation relations:

$$\begin{aligned} [L_m, L_n] &= (m - n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}C, \\ [\text{Vir}, C] &= 0, \end{aligned}$$

where  $\delta_{i,j}$  stands for Kronecker's delta. We use the following vertex operator to describe the behavior of the conformal block at a regular singular point. A Verma module  $V_\Delta$  with the highest weight  $\Delta$  of the Virasoro algebra is the induced module

$$V_\Delta = \text{Ind}_{\text{Vir}_{\geq 0}}^{\text{Vir}} \mathbb{C}|\Lambda\rangle (= U(\text{Vir}) \otimes_{\text{Vir}_{\geq 0}} \mathbb{C}|\Lambda\rangle).$$

Denote the dual Verma module by  $V_\Delta^*$  such that

$$\langle \Delta | L_0 = \Delta \langle \Delta |, \quad \langle \Delta | L_{-n} = 0 \quad (n > 0).$$

A bilinear pairing  $\langle | \rangle: V_\Delta^* \times V_\Delta \rightarrow \mathbb{C}$  is uniquely defined by

$$\begin{aligned} \langle \Delta | \cdot | \Delta \rangle &= 1, \\ \langle u | L_n \cdot | v \rangle &= \langle u | \cdot L_n | v \rangle \equiv \langle u | L_n | v \rangle \quad (n \in \mathbb{Z}), \end{aligned}$$

where  $u \in V_\Delta^*$  and  $v \in V_\Delta$ .

**Definition 2.1.** Let an operator  $\Phi_{\Delta_3, \Delta_1}^{\Delta_2}(z): V_{\Delta_1} \rightarrow V_{\Delta_3}$  be defined by

$$(3) \quad [L_n, \Phi_{\Delta_3, \Delta_1}^{\Delta_2}(z)] = z^n \left( z \frac{\partial}{\partial z} + (n+1)\Delta_2 \right) \Phi_{\Delta_3, \Delta_1}^{\Delta_2}(z),$$

$$(4) \quad \Phi_{\Delta_3, \Delta_1}^{\Delta_2}(z)|\Delta_1\rangle = z^{\Delta_3 - \Delta_2 - \Delta_1} \sum_{m \in \mathbb{Z}_{\geq 0}} v_m z^m,$$

where  $v_m \in V_{\Delta_3}$  and  $v_0 = |\Delta_3\rangle$ .

We call  $\Phi_{\Delta_3, \Delta_1}^{\Delta_2}(z)$  a vertex operator. For  $u \in V_{\Delta_1}$ , since

$$\begin{aligned} &\Phi_{\Delta_3, \Delta_1}^{\Delta_2}(z)L_n u \\ &= [\Phi_{\Delta_3, \Delta_1}^{\Delta_2}(z), L_n]u + L_n \Phi_{\Delta_3, \Delta_1}^{\Delta_2}(z)u \end{aligned}$$

$$= -z^n \left( z \frac{\partial}{\partial z} + (n+1)\Delta_2 \right) \Phi_{\Delta_3, \Delta_1}^{\Delta_2}(z)u + L_n \Phi_{\Delta_3, \Delta_1}^{\Delta_2}(z)u,$$

we only need to determine the action of the vertex operator  $\Phi_{\Delta_3, \Delta_1}^{\Delta_2}(z)$  on the highest weight  $|\Delta_1\rangle$ , namely,  $v_m$  for  $m \geq 1$ . The relation (3) on  $|\Delta_1\rangle$  for  $n \geq 0$  is equal to

$$(5) \quad L_n v_m = (\Delta_3 + n\Delta_2 - \Delta_1 + m - n + \delta_{n,0}\Delta_1) v_{m-n}.$$

Considering the case of  $n = 0$ , we see  $v_m \in V_m = \{v \in V_{\Delta_3} | L_0 v = (\Delta_3 + m)v\}$  ( $m \geq 0$ ). A basis of the eigenspace  $V_m$  of  $L_0$  is  $\{L_{-\lambda}|\Delta_3\rangle | |\lambda| = m\}$ , where  $\lambda = (\lambda_1, \dots, \lambda_k)$  ( $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$ ) is a partition of a positive integer,  $|\lambda| = \sum_{i=1}^k \lambda_i$  and  $L_{-\lambda} = L_{-\lambda_1} \cdots L_{-\lambda_k}$ . Set  $v_m = \sum_{\lambda, |\lambda|=m} c_\lambda L_{-\lambda}|\Delta_3\rangle$ . We have

$$(\langle \Delta_3 | L_\lambda v_m \rangle)_{|\lambda|=m} = (\langle \Delta_3 | L_\lambda L_{-\mu} |\Delta_3\rangle)_{|\lambda|=|\mu|=m} \cdot (c_\mu)_{|\mu|=m},$$

where  $L_\lambda = L_{\lambda_k} \cdots L_{\lambda_1}$ . Hence, if the Kac determinant  $|\langle \Delta_3 | L_\lambda L_{-\mu} |\Delta_3\rangle|_{|\lambda|=|\mu|=m}$  is not zero, then the coefficients  $c_\mu$  ( $|\mu| = m$ ) are uniquely determined recursively by (5). It follows from the construction that  $v_m$  ( $m \geq 0$ ) satisfy the relation (5). Since the condition that all Kac determinants are not zero, is equivalent to irreducibility of a Verma module, we have the following proposition.

**Proposition 2.2.** *If the Verma module  $V_{\Delta_3}$  is irreducible, then the vertex operator  $\Phi_{\Delta_3, \Delta_1}^{\Delta_2}(z)$  exists uniquely.*

We define the  $n$ -point conformal block as the expectation value of the vertex operators:

$$\langle \Delta_n | \cdot \left( \Phi_{\Delta_n, \Delta_{n-3}}^{\Delta_{n-1}}(z_{n-1}) \circ \cdots \circ \Phi_{\Delta_1, \Delta_1}^{\Delta_2}(z_2) | \Delta_1 \rangle \right),$$

where we suppose the Verma modules  $V_{\tilde{\Delta}_i}$  ( $i = 1, \dots, n-2$ ) and  $V_{\Delta_n}$  are irreducible. These conformal blocks were introduced in [4], are formal power series in  $z_i/z_{i+1}$  ( $i = 2, \dots, n-2$ ) and are believed to be absolutely convergent in the domain  $|z_2| < \cdots < |z_{n-1}|$ . As an example, let us see the first part of the four-point conformal block

$$\begin{aligned} & \langle \Delta_4 | \cdot \left( \Phi_{\Delta_4, \Delta}^{\Delta_3}(z_3) \circ \Phi_{\Delta, \Delta_1}^{\Delta_2}(z_2) | \Delta_1 \rangle \right) \\ &= z_3^{\Delta_4 - \Delta_3 - \Delta_2} z_2^{\Delta - \Delta_2 - \Delta_1} \left( 1 + \frac{(\Delta + \Delta_3 - \Delta_4)(\Delta + \Delta_2 - \Delta_1)}{2\Delta} \frac{z_2}{z_3} + \sum_{k=2}^{\infty} c_k \left( \frac{z_2}{z_3} \right)^k \right). \end{aligned}$$

The first term  $c_1$  is factorized but the second term  $c_2$  is already complicated, and we had not known an explicit formula for coefficients  $c_k$  until recently. In 2009, Alday, Gaiotto and Tachikawa conjectured a correspondence between Virasoro conformal blocks of 2d Liouville CFT and the Nekrasov partition function of a certain class of 4d supersymmetric gauge theories [2]. This correspondence was proved by Alba, Fateev, Litvinov and Tarnopolskiy [1]. We present the explicit series representation of the four-point conformal block for the central charge  $c = 1$ . Put

$$\Delta_1 = \theta_0^2, \quad \Delta_2 = \theta_t^2, \quad \Delta = \sigma^2, \quad \Delta_3 = \theta_1^2, \quad \Delta_4 = \theta_\infty^2, \quad z_2 = t, \quad z_3 = 1.$$

Then, by AGT correspondence, we have

$$(6) \quad \langle \theta_\infty^2 | \cdot \left( \Phi_{\theta_\infty^2, \sigma^2}^{\theta_1^2}(1) \circ \Phi_{\sigma^2, \theta_0^2}^{\theta_t^2}(t) | \theta_0^2 \rangle \right) = t^{\sigma^2 - \theta_0^2 - \theta_t^2} (1-t)^{2\theta_0\theta_1} \sum_{\lambda, \mu \in \mathbb{Y}} N_{\lambda, \mu} \begin{pmatrix} \theta_1, \theta_t \\ \theta_\infty, \sigma, \theta_0 \end{pmatrix} t^{|\lambda|+|\mu|},$$

where  $\mathbb{Y}$  stands for the set of all Young diagrams,

$$N_{\lambda, \mu} \left( \begin{matrix} \theta_1, \theta_t \\ \theta_\infty, \sigma, \theta_0 \end{matrix} \right) = \prod_{(i,j) \in \lambda} \frac{((\theta_t + \sigma + i - j)^2 - \theta_0^2)((\theta_1 + \sigma + i - j)^2 - \theta_\infty^2)}{h_\lambda^2(i, j)(\lambda'_j + \mu_i - i - j + 1 + 2\sigma)^2} \\ \times \prod_{(i,j) \in \mu} \frac{((\theta_t - \sigma + i - j)^2 - \theta_0^2)((\theta_1 - \sigma + i - j)^2 - \theta_\infty^2)}{h_\mu^2(i, j)(\mu'_j + \lambda_i - i - j + 1 - 2\sigma)^2},$$

and  $\lambda = (\lambda_1, \dots, \lambda_n)$  ( $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0$ ),  $|\lambda| = \sum_{i=1}^n \lambda_i$ ,  $\lambda'$  denotes the transposition of  $\lambda$ , and  $h_\lambda(i, j)$  is the hook length defined by  $h_\lambda(i, j) = \lambda_i + \lambda'_j - i - j + 1$ . We note that the above series expression is a generalization of Gauss's hypergeometric series. If we set  $\theta_t = 1/2$ , then  $\mu$  must be the empty set  $\{\phi\}$  and if  $\sigma = \theta_0 + 1/2$ , then  $\lambda$  must be  $(1^n)$  ( $n \in \mathbb{Z}_{\geq 0}$ ). Hence, when  $\theta_t = 1/2$  and  $\sigma = \theta_0 + 1/2$ , we have

$$\sum_{\lambda, \mu \in \mathbb{Y}} N_{\lambda, \mu} \left( \begin{matrix} \theta_1, 1/2 \\ \theta_\infty, \theta_0 + 1/2, \theta_0 \end{matrix} \right) t^{|\lambda|+|\mu|} = \sum_{n=0}^{\infty} t^n \prod_{i=1}^n \frac{((\theta_1 + \theta_0 + i - 1/2)^2 - \theta_\infty^2)}{n!(2\theta_0 + i)}.$$

### 3. KNOWN RESULTS OF IRREGULAR CONFORMAL BLOCKS

**3.1. Integral representation.** The Knizhnik-Zamolodchikov (KZ) equations satisfied by the correlation functions of the conformal field theory associated with an affine Lie algebra, are quantization of the Schlesinger equations [27], [15]. For  $\widehat{\mathfrak{sl}}_2$  case, solutions to KZ equations are corresponding to Virasoro conformal blocks [28], [23]. The Schlesinger equation describes isomonodromic deformation of a Fuchsian system of regular singular type. Quantizations of irregular versions of the Schlesinger equations were performed in [9], [3] for Poincaré rank one case and in [19] for any Poincaré rank case associated with  $\mathfrak{sl}_2$ . Solutions to irregular versions of KZ equations above were given by integral representations of confluent hypergeometric type.

In [22], such integral representations were constructed by free field realizations of the confluent primary fields. Since this method is applicable to the Virasoro case, we have integral formulas for irregular (Virasoro) conformal blocks. These are generalizations of integral formulas for regular conformal blocks obtained by Dotsenko and Fateev [8].

We remark that integral formulas (of Dotsenko-Fateev type) exist for particular conformal blocks only. As an example, for the four-point conformal block, integral formulas exist if and only if two parameters within five parameters are parametrized by non-negative integers, as we did it to obtain Gauss's hypergeometric series in the previous subsection. Hence, we know global representations of irregular conformal blocks for Virasoro algebra in special cases.

As an example, an integral formula of a three-point function with two regular singular points and one irregular singular point of rank one is of the form

$$\int_0^1 \cdots \int_0^1 dt_1 \cdots dt_n \prod_{1 \leq i < j \leq n} (t_i - t_j)^{2\beta} \prod_{i=1}^n t_i^u (1 - t_i)^v e^{zt_i},$$

which is a generalization of the integral representation of Kummer's confluent hypergeometric function.

**3.2. Pairings of irregular vectors embedded in Verma modules.** For general case, the following method was introduced in [11] and developed in [6], [12]. It is to take a pairing of irregular vectors  $|\Lambda\rangle$  and  $\langle\Lambda'|$  embedded in a Verma module and a dual Verma module, respectively, such that for non-negative integers  $r, s$ , and tuples  $\Lambda = (\Lambda_r, \Lambda_{r+1}, \dots, \Lambda_{2r})$ ,  $\Lambda' = (\Lambda'_s, \Lambda'_{s+1}, \dots, \Lambda'_{2s})$ ,

$$(7) \quad \begin{aligned} L_n|\Lambda\rangle &= \Lambda_n|\Lambda\rangle \quad (n = r, \dots, 2r), \quad L_n|\Lambda\rangle = 0 \quad (n > 2r), \\ \langle\Lambda'|L_n &= \Lambda'_n\langle\Lambda'| \quad (n = -s, \dots, -2s), \quad \langle\Lambda'|L_n = 0 \quad (n < -2s). \end{aligned}$$

For  $n = 0$ , the identity (7) is a well-known condition of the highest weight vector of the Verma module and is equal to the action of  $L_n$  on the primary field. For  $n > 0$ , the identities (7) are equal to the actions of  $L_n$  on the confluent primary field of rank  $r$  ((3.1) in [22]). It is natural to think that (7) is a condition of an irregular version of the highest weight vector. Hence, we expect that a pairing of such vectors produces an irregular conformal block.

When  $r = 1$ , the irregular vector  $|\Lambda\rangle$  in a Verma module of the Virasoro algebra was obtained by a degeneration limit of the image  $\Phi_{\Delta_3, \Delta_1}^{\Delta_2}(z)|\Delta_1\rangle$  of a highest weight vector by a vertex operator in [21] and given in terms of Jack symmetric functions [29]. When  $r > 1$ , the construction of irregular vectors was performed in [10] and it was revealed that the first-order irregular vectors are uniquely determined by the condition (7), up to a scalar. However, the higher-order irregular vectors contain an infinite number of parameters, and thus they are not unique. Consequently, irregular conformal blocks as pairings of irregular vectors are not uniquely determined by (7). In [12], the authors proposed a method of constructing irregular vectors by delicate limiting procedures from the image of a highest weight vector by compositions of vertex operators. Hence, if their method is justified, then we have uniquely determined irregular conformal blocks.

In what follows, we demonstrate how to take a limit of  $\Phi_{\Delta_3, \Delta_1}^{\Delta_2}(z)|\Delta_1\rangle$  as  $z$  goes to zero, following [21].

From the case of  $n = 1, 2$  of (5):

$$(8) \quad L_1 v_m = (\Delta_3 + \Delta_2 - \Delta_1 + m - 1)v_{m-1},$$

$$(9) \quad L_2 v_m = (\Delta_3 + 2\Delta_2 - \Delta_1 + m - 2)v_{m-2},$$

if we set

$$(10) \quad \Delta_2 - \Delta_1 = c_1\Lambda + c_{10} + O(\Lambda^{-1}),$$

$$(11) \quad 2\Delta_2 - \Delta_1 = c_2\Lambda^2 + c_{21}\Lambda + c_{20} + O(\Lambda^{-1}),$$

we obtain  $v_m = O(\Lambda^m)$  ( $\Lambda \rightarrow \infty$ ). Hence, when  $z$  goes to zero as  $z = w/\Lambda$  ( $\Lambda \rightarrow \infty$ ),  $v_m z^m$  becomes finite. The relations (8) and (9) transform to

$$(12) \quad L_1 p_m = c_1 p_{m-1}, \quad L_2 p_m = c_2 p_{m-2},$$

where  $v_m/\Lambda^m \rightarrow p_m$  ( $\Lambda \rightarrow \infty$ ). Put  $|W\rangle = \sum_{m=0}^{\infty} p_m w^m$ . Then, we have

$$L_1|W\rangle = c_1 w|W\rangle, \quad L_2|W\rangle = c_2 w^2|W\rangle$$

from (12).

We remark that when the central charge  $c = 1$ , the conditions (10) and (11) reduce to (13) if  $c_2 \neq 0$  and to (13) plus (15) if  $c_2 = 0$ .



**3.3. Degeneration limits of the Nekrasov partition functions.** On the gauge theory side, it is easy to compute degeneration limits of the Nekrasov partition functions. The process of the limit is called decoupling of the matter hypermultiplets in the language of the gauge theory. Let us see that by how to take successive limits of  $N_{\lambda,\mu}$ .

Looking at factors  $\theta_t \pm \sigma - \theta_0$  and  $\theta_t \pm \sigma + \theta_0$  in  $N_{\lambda,\mu}$ , we find that

$$(13) \quad \theta_1 + \theta_\infty = \Lambda, \quad \theta_1 - \theta_\infty = \theta_*, \quad t \rightarrow \frac{t}{\Lambda}, \quad \Lambda \rightarrow \infty$$

transforms  $N_{\lambda,\mu} t^{|\lambda|+|\mu|}$  to

$$N_{\lambda,\mu}(\theta_0, \theta_t, \sigma, \theta_*) t^{|\lambda|+|\mu|} = t^{|\lambda|+|\mu|} \prod_{(i,j) \in \lambda} \frac{(\theta_* + \sigma + i - j)((\theta_t + \sigma + i - j)^2 - \theta_0^2)}{h_\lambda^2(i, j)(\lambda'_j + \mu_i - i - j + 1 + 2\sigma)^2} \\ \times \prod_{(i,j) \in \mu} \frac{(\theta_* - \sigma + i - j)((\theta_t - \sigma + i - j)^2 - \theta_0^2)}{h_\mu^2(i, j)(\mu'_j + \lambda_i - i - j + 1 - 2\sigma)^2}.$$

The degenerate function  $\sum_{\lambda, \mu \in \mathbb{Y}} N_{\lambda,\mu}(\theta_0, \theta_t, \sigma, \theta_*) t^{|\lambda|+|\mu|}$  is the building block of the tau function of the fifth Painlevé function at  $t = 0$  [14].

Next, we can take two limits. One is the same limit

$$(14) \quad \theta_t + \theta_0 = \Lambda, \quad \theta_t - \theta_0 = \theta_*, \quad t \rightarrow \frac{t}{\Lambda}, \quad \Lambda \rightarrow \infty$$

as above. We get from  $N_{\lambda,\mu}(\theta_0, \theta_t, \sigma, \theta_*) t^{|\lambda|+|\mu|}$

$$N_{\lambda,\mu}(\theta_*, \sigma, \theta_*) t^{|\lambda|+|\mu|} = t^{|\lambda|+|\mu|} \prod_{(i,j) \in \lambda} \frac{(\theta_* + \sigma + i - j)(\theta_* + \sigma + i - j)}{h_\lambda^2(i, j)(\lambda'_j + \mu_i - i - j + 1 + 2\sigma)^2} \\ \times \prod_{(i,j) \in \mu} \frac{(\theta_* - \sigma + i - j)(\theta_* - \sigma + i - j)}{h_\mu^2(i, j)(\mu'_j + \lambda_i - i - j + 1 - 2\sigma)^2}.$$

The degenerate function  $\sum_{\lambda, \mu \in \mathbb{Y}} N_{\lambda,\mu}(\theta_*, \sigma, \theta_*) t^{|\lambda|+|\mu|}$  is the building block of the tau function of the third Painlevé function at  $t = 0$  [14]. Another one is that

$$(15) \quad \theta_* = \Lambda, \quad t \rightarrow \frac{t}{\Lambda}, \quad \Lambda \rightarrow \infty.$$

We get from  $N_{\lambda,\mu}(\theta_0, \theta_t, \sigma, \theta_*) t^{|\lambda|+|\mu|}$

$$N_{\lambda,\mu}(\theta_0, \theta_t, \sigma) t^{|\lambda|+|\mu|} = t^{|\lambda|+|\mu|} \prod_{(i,j) \in \lambda} \frac{((\theta_t + \sigma + i - j)^2 - \theta_0^2)}{h_\lambda^2(i, j)(\lambda'_j + \mu_i - i - j + 1 + 2\sigma)^2} \\ \times \prod_{(i,j) \in \mu} \frac{((\theta_t - \sigma + i - j)^2 - \theta_0^2)}{h_\mu^2(i, j)(\mu'_j + \lambda_i - i - j + 1 - 2\sigma)^2}.$$

If we set  $\theta_t + \theta_0 = \theta_*$  and  $\theta_t - \theta_0 = \theta_*$ , then  $N_{\lambda,\mu}(\theta_0, \theta_t, \sigma)$  coincides with  $N_{\lambda,\mu}(\theta_*, \sigma, \theta_*)$ .

Next, by the limit

$$(16) \quad \theta_* = \Lambda, \quad t \rightarrow \frac{t}{\Lambda}, \quad \Lambda \rightarrow \infty,$$

we obtain from  $N_{\lambda,\mu}(\theta_*, \sigma, \theta_*) t^{|\lambda|+|\mu|}$

$$N_{\lambda,\mu}(\theta_*, \sigma) t^{|\lambda|+|\mu|} = t^{|\lambda|+|\mu|} \prod_{(i,j) \in \lambda} \frac{(\theta_* + \sigma + i - j)}{h_\lambda^2(i, j)(\lambda'_j + \mu_i - i - j + 1 + 2\sigma)^2} \\ \times \prod_{(i,j) \in \mu} \frac{(\theta_* - \sigma + i - j)}{h_\mu^2(i, j)(\mu'_j + \lambda_i - i - j + 1 - 2\sigma)^2}.$$

The degenerate function  $\sum_{\lambda,\mu \in \mathbb{Y}} N_{\lambda,\mu}(\theta_*, \sigma) t^{|\lambda|+|\mu|}$  is the building block of the tau function of the third Painlevé function of type  $D_7^{(1)}$  at  $t = 0$  [14].

Finally, by the limit

$$(17) \quad \theta_* = \Lambda, \quad t \rightarrow \frac{t}{\Lambda}, \quad \Lambda \rightarrow \infty,$$

we obtain from  $N_{\lambda,\mu}(\theta_*, \sigma) t^{|\lambda|+|\mu|}$

$$N_{\lambda,\mu}(\sigma) t^{|\lambda|+|\mu|} = t^{|\lambda|+|\mu|} \prod_{(i,j) \in \lambda} \frac{1}{h_\lambda^2(i, j)(\lambda'_j + \mu_i - i - j + 1 + 2\sigma)^2} \\ \times \prod_{(i,j) \in \mu} \frac{1}{h_\mu^2(i, j)(\mu'_j + \lambda_i - i - j + 1 - 2\sigma)^2}.$$

The degenerate function  $\sum_{\lambda,\mu \in \mathbb{Y}} N_{\lambda,\mu}(\sigma) t^{|\lambda|+|\mu|}$  is the building block of the tau function of the third Painlevé function of type  $D_8^{(1)}$  at  $t = 0$  [14].

The limiting procedures (13), (14), (15), (16) and (17) correspond to the following degeneration scheme

$$(0, 0, 0, 0) \longrightarrow (1, 0, 0) \longrightarrow (1, 1) \longrightarrow (1/2, 1) \longrightarrow (1/2, 1/2) \\ \searrow \quad \nearrow \\ (1/2, 0, 0)$$

where  $0, 1, 1/2$  represent the ranks of irregularities.

**3.4. Rearranged expansion.** So far, local expansions of irregular conformal blocks or degeneration limits of the Nekrasov partition functions have been obtained when their expansions are at regular singular points. In general, degeneration limits of series expansions at irregular singular points are more delicate. To our knowledge, expansions at irregular singular points had been only considered in [12] before [24]. Consider

$$(18) \quad |R^{(2)}\rangle = \Phi_{\Delta_4, \Delta}^{\Delta_3}(w) \Phi_{\Delta, \Delta_1}^{\Delta_2}(z) |\Delta_1\rangle,$$

where  $\Delta_i = \alpha_i(Q - \alpha_i)$ . In what follows, we let  $w$  go to zero, while  $z$  is in a general position. Then  $|R^{(2)}\rangle$  becomes an expansion of  $z$  at the irregular singular point zero.

We already know how to take a limit of  $\Phi_{\Delta_4, \Delta}^{\Delta_3}(w)$  (see Subsection 3.2 and 3.3) but in the limit (10) and (11), the coefficients  $R_k$  of  $z^k$  in

$$|R^{(2)}\rangle = z^{\Delta - \Delta_2 - \Delta_1} w^{\Delta_4 - \Delta_3 - \Delta} \sum_{k=0}^{\infty} R_k z^k$$

diverge. Instead, Gaiotto and Tschner suggested a rearranged expansion of  $|R^{(2)}\rangle$ :

$$(19) \quad |R^{(2)}\rangle = z^{\Delta-\Delta_2-\Delta_1} \left(1 - \frac{z}{w}\right)^A \sum_{k=0}^{\infty} z^k |R_k^{(1)}\rangle$$

for some constant  $A$  in Appendix D of [12]. Note that  $|R_0^{(1)}\rangle = \Phi_{\Delta_0, \Delta}^{\Delta_1}(w)|\Delta\rangle$ . The condition of the limit (10) and (11) is now

$$(20) \quad \Delta_3 - \Delta = \frac{c_1}{\epsilon} + c_{10} + O(\epsilon), \quad 2\Delta_3 - \Delta = \frac{c_2}{\epsilon^2} + \frac{c_{21}}{\epsilon} + c_{20} + O(\epsilon) \quad (\epsilon \rightarrow 0).$$

The resulting vector  $|I^{(1)}\rangle = \lim_{\epsilon \rightarrow 0} w^{-\Delta_4+\Delta_3+\Delta} |R_0^{(1)}\rangle$  as  $w = \epsilon$  satisfies

$$L_1 |I^{(1)}\rangle = c_1 |\Lambda\rangle, \quad L_2 |I^{(1)}\rangle = c_2 |\Lambda\rangle.$$

Also  $|R_k^{(1)}\rangle$  satisfy

$$(21) \quad (L_0 - w\partial_w - \Delta_3 - \Delta) |R_k^{(1)}\rangle = k |R_k^{(1)}\rangle,$$

$$(21) \quad (L_n - w^n(w\partial_w + (n+1)\Delta_3)) |R_k^{(1)}\rangle = A \sum_{s=1}^{n-1} w^{n-s} |R_{k-s}^{(1)}\rangle$$

$$+ (A + \Delta + n\Delta_2 - \Delta_1 + k - n) |R_{k-n}^{(1)}\rangle \quad (n > 0),$$

which are derived using the commutation relations (3). It is easy to see that the coefficients of vectors in the right hand side of the recursion relations admit a limit by

$$(22) \quad A = O(\epsilon^{-1}), \quad A + \Delta - \Delta_1 = O(1), \quad \epsilon \rightarrow 0.$$

By rescaling  $|R_k^{(1)}\rangle$  as  $|\tilde{R}_k^{(1)}\rangle = w^{-\Delta_0+\Delta_1+\Delta} |R_k^{(1)}\rangle$ , the coefficient in the left hand side of the recursion relations admits a limit by (20). Hence, together with (20) and (22), we can take a limit of the above recursion relations.

Now let us see how we can take a limit of  $|\tilde{R}_1^{(1)}\rangle$ . Comparing the normal expansion and the rearranged expansion of  $|R^{(2)}\rangle$ , we obtain

$$(23) \quad |\tilde{R}_1^{(1)}\rangle = b_1 L_{-1} |\tilde{R}_0^{(1)}\rangle - \frac{b_1}{w} L_0 |\tilde{R}_0^{(1)}\rangle + \left( \frac{b_1}{w} (\Delta + \Delta_3) + \frac{A}{w} \right) |\tilde{R}_0^{(1)}\rangle,$$

where

$$b_1 = \frac{\Delta + \Delta_2 - \Delta_1}{2\Delta}.$$

It is easy to see that  $b_1$  and  $b_1/w$  have a finite limit. The final term has a finite limit if and only if

$$(24) \quad A = -\frac{\beta_1}{\epsilon} + O(1), \quad A + \Delta - \Delta_1 = -\frac{c_1 \beta_1}{2c_2} - \Delta_2 + O(\epsilon), \quad \epsilon \rightarrow 0.$$

We have observed that  $|\tilde{R}_k^{(1)}\rangle$  ( $k > 1$ ) converge without additional conditions. We remark that the limit of the recursion relation (21) under (20) and (24) is equal to (27).

The remained task is to prove that  $|\tilde{R}_k^{(1)}\rangle$  has a finite limit. In [12], it was claimed that uniqueness of elements satisfying the all resulting recursion relations (27) yields convergence of  $|\tilde{R}_k^{(1)}\rangle$ . Although the uniqueness was proved in Theorem 4.2 ([24]), we could not confirm their statement. We might need another approach and hope to report it in the near future.

## 4. DIRECT APPROACH

In the previous studies, by degeneration limits from regular conformal blocks, irregular conformal blocks have been obtained as local expansions at regular singular points or integral representations [25], [7]. There was also an attempt to obtain expansions of irregular conformal blocks at irregular singular points by collision limits [12]. However, it should be better to have a direct method to obtain irregular conformal blocks, neither by confluence process nor by asymptotic expansions of integrals.

In what follows, we present a definition of irregular versions of vertex operators and define irregular conformal blocks directly, which were proposed in [24].

For  $r \in \mathbb{Z}_{\geq 0}$ , denote the Whittaker module by  $W_{\Lambda}^{[r]}$  such that

$$L_n |\Lambda\rangle = \Lambda_n |\Lambda\rangle \quad (n = r, r+1, \dots, 2r),$$

with  $\Lambda = (\Lambda_r, \dots, \Lambda_{2r})$  and  $W_{\Lambda}^{[r]}$  is spanned by linearly independent vectors of the form

$$L_{i_1} \cdots L_{i_k} |\Lambda\rangle \quad (i_1 \leq \cdots \leq i_k < r).$$

Denote the dual Whittaker module by  $W_{\Lambda}^{*,[r]}$  such that

$$\langle \Lambda | L_n = \Lambda_n \langle \Lambda | \quad (n = -r, -r-1, \dots, -2r),$$

with  $\Lambda = (\Lambda_{-r}, \dots, \Lambda_{-2r})$  and  $W_{\Lambda}^{*,[r]}$  is spanned by linearly independent vectors of the form

$$\langle \Lambda | L_{i_1} \cdots L_{i_k} \quad (-r < i_1 \leq \cdots \leq i_k).$$

See, for example, [10], [20] on the details of Whittaker modules.

A bilinear pairing  $\langle | \rangle: W_{\Lambda'}^{*,[0]} \times W_{\Lambda}^{[1]} \rightarrow \mathbb{C}$  is uniquely defined by

$$\begin{aligned} \langle \Lambda' | \cdot | \Lambda \rangle &= 1, \\ \langle u | L_n \cdot | v \rangle &= \langle u | \cdot L_n | v \rangle \equiv \langle u | L_n | v \rangle, \end{aligned}$$

where  $\langle u | \in W_{\Lambda'}^{*,[0]}$ ,  $|v\rangle \in W_{\Lambda}^{[1]}$ . Because, if  $n > 0$ , then  $L_n$  acts on  $|\Lambda\rangle$  diagonally and if  $n \leq 0$ , then  $L_n$  acts on  $\langle \Lambda' |$  diagonally. Let  $V_0^{*,[0]}$  be the irreducible highest weight representation. Then, a bilinear pairing  $\langle | \rangle: V_0^{*,[0]} \times W_{\Lambda}^{[2]} \rightarrow \mathbb{C}$  is also uniquely defined by

$$\begin{aligned} \langle 0 | \cdot | \Lambda \rangle &= 1, \\ \langle u | L_n \cdot | v \rangle &= \langle u | \cdot L_n | v \rangle \equiv \langle u | L_n | v \rangle, \end{aligned}$$

where  $\langle u | \in V_0^{*,[0]}$ ,  $|v\rangle \in W_{\Lambda}^{[2]}$  because  $\langle 0 | L_1 = 0$ . For higher cases, one way of defining a pairing is to use the Fock space, where a pairing is defined naturally.

**Definition 4.1.** For positive integer  $r$ , we define a vertex operator  $\Phi_{\Lambda',\Lambda}^{\Delta}(z) : W_{\Lambda}^{[r]} \rightarrow W_{\Lambda'}^{[r]}$  by

$$(25) \quad [L_n, \Phi_{\Lambda',\Lambda}^{\Delta}(z)] = z^n \left( z \frac{\partial}{\partial z} + (n+1)\Delta \right) \Phi_{\Lambda',\Lambda}^{\Delta}(z),$$

$$(26) \quad \Phi_{\Lambda',\Lambda}^{\Delta}(z) |\Lambda\rangle = z^{\alpha} \exp \left( \sum_{n=0}^r \frac{\beta_n}{z^n} \right) \sum_{n=0}^{\infty} w_n z^n,$$

where  $\alpha, \beta_n \in \mathbb{C}$ ,  $w_n \in W_{\Lambda'}^{[r]}$  and  $w_0 = |\Lambda'\rangle$ .

For the case of  $r = 1$ , by the commutation relation (25) and condition (26),  $w_n$  should satisfy

$$(27) \quad (L_n - \delta_{n,1}\Lambda_1 - \delta_{n,2}\Lambda_2)w_m = -\beta_1 w_{m+1-n} + (\alpha + (n+1)\Delta + m - n)w_{m-n}$$

and

$$\alpha = -\frac{\beta_1(\Lambda_1 - \beta_1)}{2\Lambda_2} - 2\Delta_2, \quad \Lambda'_1 = \Lambda - \beta_1, \quad \Lambda'_2 = \Lambda_2.$$

For a partition  $\lambda = (\lambda_1, \dots, \lambda_n)$  ( $\lambda_i \geq \lambda_{i+1}$ ), define  $L_{-\lambda} = L_{-\lambda_1+r} \cdots L_{-\lambda_n+r}$ . By PBW theorem, the set consisting of the vectors  $L_{-\lambda}|\Lambda\rangle$  where  $\lambda$  runs over all partitions, is a basis of  $W_\Lambda^{[r]}$ . Let  $U_m$  be the subspace generated by  $L_{-\lambda}|\Lambda\rangle$  such that  $|\lambda| \leq m$ .

**Theorem 4.2** ([24]). *For any positive integer  $r$  and non-zero  $\Lambda_{2r}$ , the rank 0 vertex operator  $\Phi_{\Lambda',\Lambda}^\Delta(z): W_\Lambda^{[r]} \rightarrow W_{\Lambda'}^{[r]}$  exists and is uniquely determined by the given parameters  $\Lambda, \Delta, \beta_r$ . In particular,*

$$\Lambda'_n = \Lambda_n - \delta_{n,r}\beta_r \quad (n = r, \dots, 2r),$$

$\alpha$  and  $\beta_n$  for  $n = 1, \dots, r-1$  are polynomials in  $\Delta, \beta_r, \Lambda_r, \dots, \Lambda_{2r}, \Lambda_{2r}^{-1}$ . Moreover,  $v_m \in U_m$  and the coefficients  $c_\lambda$  of the vectors  $L_{-\lambda}|\Lambda'\rangle$  in  $v_m$  are uniquely determined as polynomials in  $\Delta, \beta_r, \Lambda_r, \dots, \Lambda_{2r}, \Lambda_{2r}^{-1}$ .

Note that by scaling the variable  $z$ , we can remove  $\Lambda_{2r}^{-1}$  in the coefficients. As a result, the coefficients of  $\Phi_{\Lambda',\Lambda}^\Delta(z)|\Lambda\rangle$  are polynomials in  $\Delta, \beta_r, \Lambda_r, \dots, \Lambda_{2r}$ . In contrast, the coefficients of  $\Phi_{\Delta_3,\Delta_1}^{\Delta_2}(z)|\Delta_1\rangle$  are rational functions in  $\Delta_3$ .

Irregular conformal blocks are defined as expectation values of irregular vertex operators in the same way of regular conformal blocks. A three-point irregular conformal block with two regular singular points  $z, \infty$  and one irregular singular point 0 of rank 1 is defined by

$$(28) \quad \left( \langle \Delta | \Phi_{\Delta,\Delta_1}^{*,\Delta_2}(z) \rangle \right) \cdot |\Lambda\rangle,$$

or

$$(29) \quad \langle \Delta | \cdot \left( \Phi_{\Lambda',\Lambda}^{\Delta_2}(z) |\Lambda\rangle \right),$$

where  $\Phi_{\Delta,\Delta_1}^{*,\Delta_2}(z): V_\Delta^* \rightarrow V_{\Delta_1}^*$  is the dual vertex operator and  $\Lambda = (\Lambda_1, \Lambda_2)$  ( $\Lambda_2 \neq 0$ ). Although for the regular four-point case, the following two conformal blocks

$$\left( \langle \Delta_4 | \Phi_{\Delta_4,\Delta}^{*,\Delta_3}(z_3) \rangle \right) \cdot \left( \Phi_{\Delta,\Delta_1}^{\Delta_2}(z_2) |\Delta_1\rangle \right), \quad \langle \Delta_4 | \cdot \left( \Phi_{\Delta_4,\Delta}^{\Delta_3}(z_3) |\Delta_1\rangle \circ \Phi_{\Delta,\Delta_1}^{\Delta_2}(z_2) |\Delta_1\rangle \right)$$

are equal, the three-point irregular conformal blocks (28), (29) are different. The former is an expansion at the regular singular point  $\infty$  and the latter is an expansion at the irregular singular point 0.

A two-point irregular conformal block with one regular singular point  $z$  and one irregular singular point 0 of rank 2 is defined by

$$(30) \quad \langle 0 | \cdot \left( \Phi_{\Lambda',\Lambda}^{\Delta_2}(z) |\Lambda\rangle \right),$$

where  $\Lambda = (\Lambda_2, \Lambda_3, \Lambda_4)$  ( $\Lambda_4 \neq 0$ ).

## 5. CONJECTURES

**5.1. Series expansion formulas of  $\tau_V(t)$  and  $\tau_{IV}(t)$ .** In this section, we assume the central charge  $c = 1$ . As the tau function of  $P_{VI}$  is expressed as a Fourier expansion (2) in terms of a four-point regular conformal block, we expect that expansions at the irregular singular point  $\infty$  of the tau functions of  $P_V$ ,  $P_{IV}$  are Fourier expansions in terms of a three-point irregular conformal block with two regular singular points  $z$ ,  $\infty$  and one irregular singular point 0 of rank 1 (29), a two-point irregular conformal block with one regular singular point  $z$  and one irregular singular point 0 of rank 2 (30), respectively.

Recall that the Hamiltonian functions  $H_J(t)$  of  $P_J$  ( $J = V, IV$ ) satisfy

$$(31) \quad (th_V'')^2 - (h_V - th_V' + 2(h_V')^2)^2 + \frac{1}{4}((2h_V' - \theta)^2 - 4\theta_0^2)((2h_V' + \theta)^2 - 4\theta_t^2) = 0,$$

$$(32) \quad (H_{IV}'')^2 - 4(tH_{IV}' - H_{IV})^2 + 4H_{IV}'(H_{IV}' - 2(\theta + \theta_t))(H_{IV}' - 4\theta_t) = 0,$$

where  $f' = df/dt$  and  $h_V = tH_V$ . The  $\tau$ -functions  $\tau_J = \tau_J(t)$  are related to the Hamiltonian functions as

$$H_{IV}(t) = \frac{d}{dt} \log \tau_{IV}(t), \quad H_V(t) = t \frac{d}{dt} \log \tau_V(t).$$

Note that the tau functions satisfying these differential equations define each Painlevé functions.

Based on the expectations, we substitute

$$\sum_{n \in \mathbb{Z}} s^n C_n \mathcal{F}(\theta, \Lambda, \beta_n; t),$$

where

$$\mathcal{F}(\theta, \Lambda, \beta_n; t) = \langle \theta_0^2 | \cdot \left( \Phi_{(\Lambda_1 - \beta_n, \Lambda_2), (\Lambda_1, \Lambda_2)}^{\theta_t^2}(t^{-1}) | (\Lambda_1, \Lambda_2) \rangle \right)$$

for  $P_V$  case, into (31) and

$$\langle 0 | \cdot \left( \Phi_{(\Lambda_2 - \beta_n, \Lambda_3, \Lambda_4), (\Lambda_2, \Lambda_3, \Lambda_4)}^{\theta_t^2}(t^{-1}) | (\Lambda_2, \Lambda_3, \Lambda_4) \rangle \right)$$

for  $P_{IV}$  case, into (32). Then, we look at the coefficient of  $s^i$  for  $i \in \mathbb{Z}$  which is of the form

$$t^A e^B (a_0 + a_1 t^{-1} + a_2 t^{-2} + \dots).$$

Since  $a_i$  ( $i = 0, 1, 2, \dots$ ) are polynomials in the parameters, we can determine  $C_n$ ,  $\beta_n$ ,  $\Lambda_i$  in terms of  $\theta_i$  and  $\beta$ .

**Conjecture 5.1** ( $P_V$  case, [24]). *Let*

$$\begin{aligned} \tau(t) &= \sum_{n \in \mathbb{Z}} s^n (-1)^{n(n+1)/2} G(1 \pm \theta_0 + \theta - \beta - n) G(1 + \theta_t \pm (\beta + n)) \\ &\quad \times \langle \theta_0^2 | \cdot \left( \Phi_{(\theta - \beta - n, 1/4), (\theta, 1/4)}^{\theta_t^2}(t^{-1}) | (\theta, 1/4) \rangle \right) \end{aligned}$$

and  $H = t(\log(t^{-2\theta_t^2 - \theta^2/2} e^{-\theta t/2} \tau(t)))'$ . Then,  $H$  satisfies the differential equation (31).

**Conjecture 5.2** ( $P_{IV}$  case, [24]). *Let*

$$\tau(t) = t^{-2\theta_t^2} e^{\theta t^2} \sum_{n \in \mathbb{Z}} s^n G(1 + \theta - \beta - n) \prod_{\epsilon = \pm 1} G(1 + \theta_t + \epsilon(\beta + n))$$

$$\times \langle 0 | \cdot \left( \Phi_{(\theta,0,1/4),(\theta-\beta-n,0,1/4)}^{\theta_t^2} (1/\sqrt{2}t) | (\theta, 0, 1/4) \rangle \right)$$

and  $H = (\log \tau(t))'$ . Then,  $H$  satisfies the differential equation (32).

Here,

$$\begin{aligned} (33) \quad & \langle \theta_0^2 | \cdot \left( \Phi_{(\theta,1/4),(\theta-\beta,1/4)}^{\theta_t^2} (1/t) | (\theta, 1/4) \rangle \right) \\ &= t^{2\theta_t^2+2\beta(\theta-\beta)} e^{\beta t} \left( 1 + 2 \left( 2\beta^3 - 3\beta^2\theta + \beta\theta^2 - \beta\theta_0^2 - \beta\theta_t^2 + \theta\theta_t^2 \right) t^{-1} \right. \\ &+ 2 \left( 4\beta^6 - 12\beta^5\theta + 13\beta^4\theta^2 - 4\beta^4\theta_0^2 - 4\beta^4\theta_t^2 + 5\beta^4 - 6\beta^3\theta^3 + 6\beta^3\theta\theta_0^2 + 10\beta^3\theta\theta_t^2 - 10\beta^3\theta + \beta^2\theta^4 \right. \\ &- 2\beta^2\theta^2\theta_0^2 - 8\beta^2\theta^2\theta_t^2 + 6\beta^2\theta^2 + \beta^2\theta_0^4 + 2\beta^2\theta_0^2\theta_t^2 - 3\beta^2\theta_0^2 + \beta^2\theta_t^4 - 3\beta^2\theta_t^2 + 2\beta\theta^3\theta_t^2 - \beta\theta^3 \\ &\left. \left. - 2\beta\theta\theta_0^2\theta_t^2 + \beta\theta\theta_0^2 - 2\beta\theta\theta_t^4 + 5\beta\theta\theta_t^2 + \theta^2\theta_t^4 - 2\theta^2\theta_t^2 + \theta_0^2\theta_t^2 \right) t^{-2} + \dots \right) \end{aligned}$$

and

$$\begin{aligned} & \langle 0 | \cdot \left( \Phi_{(\theta,0,1/4),(\theta-\beta,0,1/4)}^{\theta_t^2} (1/t) | (\theta, 0, 1/4) \rangle \right) \\ &= t^{3\theta_t^2+\beta(2\theta-3\beta)} e^{\beta t^2/2} \left( 1 + \left( \theta^2\beta + 2\theta\theta_t^2 - 6\theta\beta^2 - 3\theta_t^2\beta + 6\beta^3 \right) t^{-2} \right. \\ &+ \frac{1}{4} \left( 2\theta^4\beta^2 + 8\theta^3\theta_t^2\beta - 24\theta^3\beta^3 - 4\theta^3\beta + 8\theta^2\theta_t^4 - 60\theta^2\theta_t^2\beta^2 - 16\theta^2\theta_t^2 + 96\theta^2\beta^4 \right. \\ &+ 48\theta^2\beta^2 - 24\theta\theta_t^4\beta + 120\theta\theta_t^2\beta^3 + 72\theta\theta_t^2\beta - 144\theta\beta^5 - 140\theta\beta^3 - 2\theta\beta + 18\theta_t^4\beta^2 \\ &\left. \left. + \theta_t^4 - 72\theta_t^2\beta^4 - 66\theta_t^2\beta^2 - \theta_t^2 + 72\beta^6 + 105\beta^4 + 3\beta^2 \right) t^{-4} + \dots \right). \end{aligned}$$

**5.2. A combinatorial expansion formula 3-pt ICB at an irregular singular point.** It is natural to expect that these irregular conformal blocks have combinatorial expressions, as expansions of conformal blocks at regular singular points have explicit formulas parametrized by partitions. To our knowledge, in general case, such combinatorial formulas for expansions of irregular conformal blocks at irregular singular points had not been reported.

In what follows, we only consider the irregular conformal block of rank one for  $P_V$ . Note that a particular three-point irregular conformal block of rank one is the confluent hypergeometric function, whose coefficients of  $t^{-k}$  are factorized. Let us substitute  $\beta$  into  $\theta_t$  in (33). Then, the coefficient of  $t^{-1}$  becomes  $2\beta((\theta - \beta)^2 - \theta_0^2)$ . If we substitute  $\theta - \beta$  into  $\theta_0$ , we see that the coefficient of  $t^{-1}$  becomes  $2(\beta - \theta)(\beta^2 - \theta_t^2)$ . Fortunately, we have

$$2(2\beta^3 - 3\beta^2\theta + \beta\theta^2 - \beta\theta_0^2 - \beta\theta_t^2 + \theta\theta_t^2) = 2(\beta - \theta)(\beta^2 - \theta_t^2) + 2\beta((\theta - \beta)^2 - \theta_0^2).$$

Thus, the coefficient of  $t^{-1}$  is successfully written as a sum of two factorized forms, which should be corresponding to pairs of partitions  $((1), \emptyset), (\emptyset, (1))$ . These two factors look like same as the factors in the partition functions in Section 3.3. We can guess that a pair  $(\lambda, \emptyset)$  of partitions is corresponding to

$$N_{\lambda, \emptyset} = \prod_{(i,j) \in \lambda} \frac{(2(\beta - \theta) + i - j)((\beta + i - j)^2 - \theta_t^2)}{h_\lambda(i, j)^2}$$

and a pair  $(\emptyset, \mu)$  of partitions is corresponding to

$$N_{\emptyset, \mu} = (-1)^{|\mu|} \prod_{(i,j) \in \mu} \frac{(-2\beta + i - j)((\theta - \beta + i - j)^2 - \theta_0^2)}{h_\mu(i, j)^2}.$$

Let us check that this assumption works for the coefficient of  $t^{-2}$  which should be expressed by a sum of five factors corresponding to pairs of partitions  $((2), \emptyset)$ ,  $((1, 1), \emptyset)$ ,  $((1), (1))$ ,  $(\emptyset, (2))$ ,  $(\emptyset, (1, 1))$ . We see that the coefficient of  $t^{-2}$  is equal to

$$\sum_{|\lambda|=2} (N_{\lambda, \emptyset} + N_{\emptyset, \lambda}) + 2(2(\theta - \beta)\beta - 1)(\beta^2 - \theta_t^2)((\theta - \beta)^2 - \theta_0^2).$$

The last term should be corresponding to  $((1), (1))$  but contains a quadratic form which is not factored by linear equations. If we use  $N_{(1), (1)}$ , then the last term is expressed as

$$N_{(1), (1)} - 2(\beta^2 - \theta_t^2)((\theta - \beta)^2 - \theta_0^2).$$

Hence, the term corresponding to  $((1), (1))$  is written by a sum of two factored forms. Keeping this in mind, we put

$$\begin{aligned} U_\lambda &= \prod_{(i,j) \in \lambda} (2(\beta - \theta) + i - j), \quad V_\lambda = \prod_{(i,j) \in \lambda} (-2\beta + i - j), \\ S_{\lambda, \mu} &= (-1)^{|\mu|} \prod_{(i,j) \in \lambda} \frac{(\beta + i - j)^2 - \theta_t^2}{h_\lambda(i, j)^2} \prod_{(i,j) \in \mu} \frac{(\theta - \beta + i - j)^2 - \theta_0^2}{h_\mu(i, j)^2}. \end{aligned}$$

Then, the coefficient of  $t^{-3}$  is equal to

$$\sum_{|\lambda|=3} (N_{\lambda, \emptyset} + N_{\emptyset, \lambda}) + \sum_{|\lambda|=2} (N_{\lambda, (1)} - 4U_{\lambda/(1)}S_{\lambda, (1)} + N_{(1), \lambda} - 4V_{\lambda/(1)}S_{(1), \lambda}),$$

where,  $U_{\lambda/\mu} = U_\lambda/U_\mu$  and  $V_{\lambda/\mu} = V_\lambda/V_\mu$ . Similarly, the coefficient of  $t^{-4}$  is expressed as

$$\begin{aligned} &\sum_{|\lambda|=4} (N_{\lambda, \emptyset} + N_{\emptyset, \lambda}) + \sum_{|\lambda|=3} (N_{\lambda, (1)} - 6U_{\lambda/(1)}S_{\lambda, (1)} + N_{(1), \lambda} - 6V_{\lambda/(1)}S_{(1), \lambda}) \\ &+ N_{(2), (2)} - 8U_{(2)/(1)}V_{(2)/(1)}S_{(2), (2)} + 4S_{(2), (2)} \\ &+ N_{(2), (1, 1)} - 8U_{(2)/(1)}V_{(1, 1)/(1)}S_{(2), (1, 1)} + 12S_{(2), (1, 1)} \\ &+ N_{(1, 1), (2)} - 8U_{(1, 1)/(1)}V_{(2)/(1)}S_{(1, 1), (2)} + 12S_{(1, 1), (2)} \\ &+ N_{(1, 1), (1, 1)} - 8U_{(1, 1)/(1)}V_{(1, 1)/(1)}S_{(1, 1), (1, 1)} + 4S_{(1, 1), (1, 1)}. \end{aligned}$$

Note that  $N_{\lambda, \mu} = U_{\lambda/\emptyset}V_{\mu/\emptyset}S_{\lambda, \mu}$ . Based on the observations, we propose the next conjecture.

**Conjecture 5.3.** *A three-point irregular conformal block with two regular singular points  $z, \infty$  and one irregular singular point  $0$  of rank one admits the following combinatorial formula*

$$\begin{aligned} &\langle \theta_0^2 | \cdot \left( \Phi_{(\theta, 1/4), (\theta - \beta, 1/4)}^{\theta_t^2}(t) | (\theta, 1/4) \rangle \right) \\ &= t^{-2\theta_t^2 - 2\beta(\theta - \beta)} e^{\frac{\beta}{t}} \sum_{\lambda, \mu \in \mathbb{Y}} t^{|\lambda| + |\mu|} \sum_{\substack{\nu \subset \lambda, \eta \subset \mu, \\ |\nu| = |\eta|}} (-1)^{|\nu|} c_{\lambda, \mu}^{\nu, \eta} U_{\lambda/\nu} V_{\mu/\eta} S_{\lambda, \mu}, \end{aligned}$$

where  $c_{\lambda, \mu}^{\nu, \eta} \in \mathbb{Z}_{\geq 0}$ , as an expansion at the irregular singular point  $0$ .



As long as we calculate, the coefficients  $c_{\lambda,\mu}^{\nu,\eta}$  are non-negative integers. Further, we observe

$$c_{\lambda,\mu}^{\emptyset,\emptyset} = 1, \quad c_{\lambda,\mu}^{(1),(1)} = 2|\lambda||\mu|, \quad c_{\lambda,\mu}^{(2),(2)} = q_\lambda q_\mu, \quad c_{\lambda,\mu}^{(2),(1,1)} = 3q_\lambda q_{\mu'}, \quad c_{\lambda,\mu}^{\nu,\eta} = c_{\mu,\lambda}^{\eta,\nu} = c_{\lambda',\mu'}^{\nu',\eta'},$$

where

$$q_\lambda = \lambda_1(\lambda_1 - 1) + \sum_{\substack{(i,j) \in \lambda, \\ i \neq 1}} \left( \lambda_1 - 1 + \sum_{k=1}^{j-1} \lambda'_k \right).$$

The function  $q_\lambda$  of partitions can be regarded as a function of particular (column) semi-standard tableaux such that for  $\lambda$ ,  $(1, j)$  box has  $2(j-1)$  and  $(i, j)$  box for  $i \neq 1$  has  $\lambda_1 - 1 + \sum_{k=1}^{j-1} \lambda'_k$ . As an example, for  $\lambda = (6, 4, 4, 3, 2, 1)$ , the corresponding tableau is

0	2	4	6	8	10
5	11	16	20		
5	11	16	20		
5	11	16			
5	11				
5					

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